

# EXPLICIT RELATIONS BETWEEN PRIMES IN SHORT INTERVALS AND EXPONENTIAL SUMS OVER PRIMES

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**ABSTRACT.** Under the assumption of the Riemann Hypothesis (RH), we prove explicit quantitative relations between hypothetical error terms in the asymptotic formulae for truncated mean-square average of exponential sums over primes and in the mean-square of primes in short intervals. We also remark that such relations are connected with a more precise form of Montgomery's pair-correlation conjecture.

## 1. INTRODUCTION

In many circle method applications a key role is played by the asymptotic behavior as  $X \rightarrow \infty$  of the truncated mean square of the exponential sum over primes, i.e. by

$$R(X, \xi) = \int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 d\alpha, \quad \frac{1}{2X} \leq \xi \leq \frac{1}{2},$$

where  $S(\alpha) = \sum_{n \leq X} \Lambda(n) e(n\alpha)$ ,  $T(\alpha) = \sum_{n \leq X} e(n\alpha)$ ,  $e(x) = e^{2\pi i x}$  and  $\Lambda(n)$  is the von Mangoldt function. In 2000 the first author and Perelli [6] studied how to connect, under the assumption of the Riemann Hypothesis (RH) and of Montgomery's pair-correlation conjecture, the behaviour as  $X \rightarrow \infty$  of  $R(X, \xi)$  with the one of the mean-square of primes in short intervals, i.e., with

$$J(X, h) = \int_1^X (\psi(x+h) - \psi(x) - h)^2 dx, \quad 1 \leq h \leq X,$$

where  $\psi(x) = \sum_{n \leq x} \Lambda(n)$ . Recalling that Goldston and Montgomery [2] proved that the asymptotic behavior of  $J(X, h)$  as  $X \rightarrow \infty$  is related with Montgomery's pair correlation function

$$F(X, T) = 4 \sum_{0 < \gamma, \gamma' \leq T} \frac{X^{i(\gamma - \gamma')}}{4 + (\gamma - \gamma')^2},$$

where  $\gamma, \gamma'$  run over the imaginary part of the non-trivial zeros of the Riemann zeta function, the following result was proved in [6].

**Theorem.** Assume RH. As  $X \rightarrow \infty$ , the following statements are equivalent:

- (i) for every  $\varepsilon > 0$ ,  $R(X, \xi) \sim 2X\xi \log X\xi$  uniformly for  $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$ ;
- (ii) for every  $\varepsilon > 0$ ,  $J(X, h) \sim hX \log(X/h)$  uniformly for  $1 \leq h \leq X^{1/2-\varepsilon}$ ;
- (iii) for every  $\varepsilon > 0$  and  $A \geq 1$ ,  $F(X, T) \sim (T/2\pi) \log \min(X, T)$  uniformly for  $X^{1/2+\varepsilon} \leq T \leq X^A$ .

We remark that the uniformity ranges here are smaller than the ones in [2] and that it is due to the presence of  $E(X, h)$  (a term which naturally comes from Gallagher's lemma), see (7) and Lemma 3.

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In 2003 Chan [1] formulated a more precise pair-correlation hypothesis and gave explicit results for the connections between the error terms in the asymptotic formulae for  $F(X, T)$  and  $J(X, h)$ . Such results were recently extended and improved by the authors of this paper in a joint work with Perelli [7]: writing

$$F(X, T) = \frac{T}{2\pi} \left( \log \frac{T}{2\pi} - 1 \right) + R_F(X, T), \quad (1)$$

$$J(X, h) = hX \left( \log \frac{X}{h} + c' \right) + R_J(X, h) \quad (2)$$

and  $c' = -\gamma - \log(2\pi)$  ( $\gamma$  is Euler's constant), they gave explicit relations between (1) and (2) with error terms essentially of type

$$R_F(X, T) \ll \frac{T^{1-a}}{(\log T)^b} \quad \text{and} \quad R_J(X, h) \ll \frac{hX}{(\log X)^b} \left( \frac{h}{X} \right)^a,$$

with  $X, T$  and  $h$  in suitable ranges and  $a, b \geq 0$ .

Our aim here is to prove explicit connections between the error terms in the asymptotic formulae for  $R(X, \xi)$  and  $J(X, h)$  in the same fashion of [7], but, recalling the previously cited theorem in [6], we have to restrict our attention to the range  $1 \leq h \leq X^{1/2-\varepsilon}$  (or, equivalently, to  $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$ ). In what follows the implicit constants may depend on  $a, b$ . Our first result is

**Theorem 1.** *Assume RH and let  $1 \leq h \leq X^{1/2-\varepsilon}$ ,  $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$ . Let further  $0 \leq a < 1$ ,  $b \geq 0$ ,  $(a, b) \neq (0, 0)$  be fixed. If, for some constant  $c \in \mathbb{R}$ , we have*

$$R(X, \xi) = 2X\xi \log X\xi + cX\xi + \mathcal{O}\left(\frac{(X\xi)^{1-a}}{(\log X\xi)^b}\right), \quad (3)$$

then

$$J(X, h) = hX \left( \log \frac{X}{h} + c' \right) + \mathcal{O}(X + E(X, h) + R_{a,b}(X, h)),$$

provided that (3) holds uniformly for

$$\frac{1}{h} \left( \frac{h}{X} \right)^a (\log X)^{-b-4} \leq \xi \leq \frac{1}{h} \left( \frac{X}{h} \right)^a (\log X)^{b+4}, \quad (4)$$

where

$$c' = c/2 + 2 - \gamma - \log(2\pi), \quad (5)$$

$$R_{a,b}(X, h) = \begin{cases} hX \log \log X (\log X)^{-b} & \text{if } a = 0 \\ hX (h/X)^a (\log X)^{-b} & \text{if } a > 0, \end{cases} \quad (6)$$

and, for every fixed  $\varepsilon > 0$ , we define

$$E(X, h) = \begin{cases} (h+1)^3 (\log X)^2 & (\text{uncond.}) \text{ uniformly for } 0 < h \leq X^\varepsilon \\ h^3 & (\text{uncond.}) \text{ uniformly for } X^\varepsilon \leq h \leq X \\ (h+1)X (\log X)^4 & (\text{under RH}) \text{ uniformly for } 0 < h \leq X. \end{cases} \quad (7)$$

We explicitly remark that, since  $c' = -\gamma - \log(2\pi)$ , by (5) we get  $c = -4$  and that the conditions  $\xi \leq 1/2$  and (4) imply

$$h \gg X^{a/(a+1)} (\log X)^{(b+4)/(a+1)}$$

which also leads to  $R_{a,b}(X, h) \gg X$ . It is also useful to remark that the  $E(X, h) \ll R_{a,b}(X, h)$  only for  $h \ll X^{(1-a)/(2+a)} (\log X)^{-b/(2+a)}$ .

The technique used to prove Theorem 1 is similar to the one in Lemma 2 in [7]; the main difference is in the presence of the terms  $E(X, h)$  (which comes from Lemma 3) and  $\mathcal{O}(X)$  (which comes from the term  $\mathcal{O}(1)$  in Lemma 1).

Concerning the opposite direction, we have

**Theorem 2.** Assume RH and let  $1 \leq h \leq X^{1/2-\varepsilon}$ ,  $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$ . Let further  $0 \leq a < 1$ ,  $b \geq 0$ ,  $(a, b) \neq (0, 0)$  be fixed. If, for some constant  $c' \in \mathbb{R}$ , we have

$$J(X, h) = hX \left( \log \frac{X}{h} + c' \right) + \mathcal{O} \left( \frac{hX}{(\log X)^b} \left( \frac{h}{X} \right)^a \right), \quad (8)$$

then

$$R(X, \xi) = 2X\xi \log X\xi + cX\xi + \mathcal{O} \left( \frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}} \right), \quad (9)$$

provided that (8) holds uniformly for

$$\frac{1}{\xi} \frac{(X\xi)^{-a/(2a+6)}}{(\log X)^{(a+b+4)/(2a+6)}} \leq h \leq \frac{1}{\xi} (X\xi)^{4a/(a+3)} (\log X)^{(3a+4b+13)/(a+3)},$$

where  $c = 2(c' - 2 + \gamma + \log(2\pi))$ .

Note that for  $a = 0$  we have to take  $b > 2$  to get that the error term in (9) is  $o(X\xi)$ . The technique used to prove Theorem 2 is similar to the one in Lemma 5 of [7]; the main difference is in the use of Lemma 4 which is needed to provide pair-correlation independent estimates of the involved quantities.

We remark that results similar to Theorems 1-2 can be proved using the weighted quantities

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/X} e(n\alpha), \quad \tilde{T}(\alpha) = \sum_{n=1}^{\infty} e^{-n/X} e(n\alpha),$$

$$\tilde{R}(X, \xi) = \int_{-\xi}^{\xi} |\tilde{S}(\alpha) - \tilde{T}(\alpha)|^2 d\alpha, \quad \tilde{J}(X, h) = \int_0^{\infty} (\psi(x+h) - \psi(x) - h)^2 e^{-2x/X} dx$$

in place of  $S(\alpha)$ ,  $T(\alpha)$ ,  $R(X, \xi)$  and  $J(X, h)$ , respectively. The proofs are similar; in the analogue of Theorem 1 the main difference is in using the second part of Lemma 3 thus replacing  $E(X, h)$  with the sharper quantity  $\tilde{E}(X, h)$  defined in (17). Concerning the analogue of Theorem 2, the key point is in Eq. (39): in this case we will be able to extend its range of validity to  $\xi \leq x \leq \xi X^{1-\varepsilon}$  and to get rid of the term  $(x^3/\xi)(\log X)^2$ . These remarks lead to results which hold in wider ranges:  $1 \leq h \leq X^{1-\varepsilon}$  and  $X^{-1+\varepsilon} \leq \xi \leq 1/2$ .

The order of magnitude of  $\tilde{J}(X, h)$  can be directly deduced from the one of  $J(X, h)$  via partial integration, see *e.g.* eq. (21). Unfortunately, the vice-versa seems to be very hard to achieve; this depends on the fact that we do not have sufficiently strong Tauberian theorems to get rid of the exponential weight in the definition of  $\tilde{J}(X, h)$ . Such a phenomenon is well known in the literature, see, *e.g.*, Heath-Brown's remark on pages 385-386 of [4].

## 2. SOME LEMMAS

In the following we will need two weight functions.

**Definition 1.** For  $h > 0$  we let

$$K(\alpha, h) = \sum_{-h \leq n \leq h} (h - |n|) e(n\alpha) \quad \text{and} \quad U(\alpha, h) = \left( \frac{\sin(\pi h \alpha)}{\pi \alpha} \right)^2. \quad (10)$$

We will need some information about the total mass of such weights.

**Lemma 1.** *For  $h > 0$ , we have*

$$\int_0^{1/2} K(\alpha, h) d\alpha = \frac{h}{2} \quad \text{and} \quad \int_0^{+\infty} U(\alpha, h) d\alpha = \frac{h}{2}.$$

Moreover we also have

$$\begin{aligned} \int_0^{1/2} \log(h\alpha) K(\alpha, h) d\alpha &= -\frac{h}{2}(\log(2\pi) + \gamma - 1) + \mathcal{O}(1), \\ \int_0^{+\infty} \log(h\alpha) U(\alpha, h) d\alpha &= -\frac{h}{2}(\log(2\pi) + \gamma - 1). \end{aligned}$$

Before the proof, we remark that this lemma is consistent with the constant in Lemma 2 of Languasco, Perelli and Zaccagnini [7], taking into account the fact that our variable  $h$  here corresponds to  $\pi\kappa$  there.

*Proof.* The results on  $U(\alpha, h)$  can be immediately obtained by integrals n.3.821.9 and n.4.423.3, respectively on pages 460 and 594 of Gradshteyn and Ryzhik [3].

Now we prove the part concerning  $K(\alpha, h)$ . The first identity immediately follows by isolating the contribution of  $n = 0$  in the definition of  $K(\alpha, h)$  and making a trivial computation. To prove the second identity, separating again the contribution of the term  $n = 0$  and using standard properties of the complex exponential functions, we have

$$\begin{aligned} I(h) &:= 2 \int_0^{1/2} \log(h\alpha) K(\alpha, h) d\alpha \\ &= 2h \int_0^{1/2} \log(h\alpha) d\alpha + 4 \sum_{1 \leq n \leq h} (h - n) \int_0^{1/2} \log(h\alpha) \cos(2\pi n\alpha) d\alpha \\ &= h \log h - h(\log 2 + 1) + 2 \sum_{1 \leq n \leq h} (h - n) \int_0^1 \log\left(\frac{h\beta}{2}\right) \cos(\pi n\beta) d\beta. \end{aligned}$$

We remark that

$$\int_0^1 \log\left(\frac{h}{2}\right) \cos(\pi n\beta) d\beta = 0$$

whenever  $n$  is a positive integer, and hence we can write

$$\begin{aligned} I(h) &= h \log h - h(\log 2 + 1) + 2 \sum_{1 \leq n \leq h} (h - n) \int_0^1 \log \beta \cos(\pi n\beta) d\beta \\ &= h \log h - h(\log 2 + 1) - \sum_{1 \leq n \leq h} \frac{h - n}{n} - 2 \sum_{1 \leq n \leq h} (h - n) \frac{\text{si}(\pi n)}{\pi n}, \end{aligned}$$

by Formula 4.381.2 on page 581 of [3], where the sine integral function is defined by

$$\text{si}(x) = - \int_x^{+\infty} \frac{\sin t}{t} dt \quad (11)$$

for  $x > 0$ . The elementary relation  $\sum_{1 \leq n \leq h} 1/n = \log h + \gamma + \mathcal{O}(h^{-1})$  shows that

$$I(h) = -h(\log 2 + \gamma) + \mathcal{O}(1) - \frac{2h}{\pi} \sum_{1 \leq n \leq h} \frac{\text{si}(\pi n)}{n} + \frac{2}{\pi} \sum_{1 \leq n \leq h} \text{si}(\pi n).$$

Finally we remark that Eq. (11) implies, by means of a simple integration by parts, that  $\text{si}(x) \ll x^{-1}$  as  $x \rightarrow +\infty$ . Hence

$$\sum_{1 \leq n \leq h} \frac{\text{si}(\pi n)}{n} = \sum_{n \geq 1} \frac{\text{si}(\pi n)}{n} + \mathcal{O}(h^{-1}) = \frac{\pi}{2}(\log \pi - 1) + \mathcal{O}(h^{-1}),$$

by Formula 6.15.2 on page 154 of [9]. Moreover, by a double partial integration, we get

$$\text{si}(x) = -\frac{\cos x}{x} - \frac{\sin x}{x^2} + 2 \int_x^{+\infty} \frac{\sin t}{t^3} dt$$

and hence

$$\sum_{1 \leq n \leq h} \text{si}(\pi n) = \sum_{1 \leq n \leq h} \frac{(-1)^{n+1}}{\pi n} + \mathcal{O}\left(\sum_{1 \leq n \leq h} \frac{1}{n^2}\right) \ll 1.$$

In conclusion

$$I(h) = -h(\log 2 + \gamma) - \frac{2h}{\pi} \left( \frac{\pi}{2}(\log \pi - 1) + \mathcal{O}(h^{-1}) \right) + \mathcal{O}(1),$$

and Lemma 1 is proved.  $\square$

Now we see some information about the order of magnitude of  $K(\alpha, h)$  and its first derivative.

**Lemma 2.** *For  $h \geq 1$  we have*

$$K(\alpha, h) \ll \min(h^2, \|\alpha\|^{-2}),$$

and

$$\frac{d}{d\alpha} K(\alpha, h) \ll h \|\alpha\| \min(h^3, \|\alpha\|^{-3}).$$

*Proof.* We assume that  $\alpha \in (0, 1/2)$  as we may. We let  $h_0 = [h]$ . We first remark that

$$\begin{aligned} K(\alpha, h) &= \sum_{-h_0 \leq n \leq h_0} (h_0 - |n|) e(n\alpha) + \{h\} \sum_{-h_0 \leq n \leq h_0} e(n\alpha) \\ &= \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \right)^2 + \{h\} \sum_{-h_0 \leq n \leq h_0} e(n\alpha). \end{aligned}$$

Recalling the identity

$$\sum_{-h_0 \leq n \leq h_0} e(n\alpha) = 1 + 2 \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \cos(\pi(h_0 + 1)\alpha) \quad (12)$$

and the estimate

$$\frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \ll \min(h_0, \alpha^{-1}), \quad (13)$$

the first part of the lemma immediately follows.

For the second inequality we first remark that

$$\begin{aligned} \frac{d}{d\alpha} K(\alpha, h) &= 2\pi i \sum_{-h \leq n \leq h} (h - |n|) n e(n\alpha) \\ &= 2\pi i \sum_{-h_0 \leq n \leq h_0} (h_0 - |n|) n e(n\alpha) + 2\pi i \{h\} \sum_{-h_0 \leq n \leq h_0} n e(n\alpha) \\ &= A(\alpha, h) + B(\alpha, h), \end{aligned}$$

say. By (12) we get that

$$A(\alpha, h) = \frac{d}{d\alpha} \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \right)^2 \quad \text{and} \quad B(\alpha, h) = 2\{h\} \frac{d}{d\alpha} \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \cos(\pi(h_0 + 1)\alpha) \right),$$

respectively. We remark that

$$\frac{d}{d\alpha} \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} = \frac{\pi h_0 \cos(\pi h_0 \alpha) \sin(\pi \alpha) - \pi \sin(\pi h_0 \alpha) \cos(\pi \alpha)}{(\sin(\pi \alpha))^2}. \quad (14)$$

For  $\alpha \leq h_0^{-1}$  a standard development shows that the numerator in (14) is  $\ll \alpha^3 h_0^3$ , while the denominator is  $\gg \alpha^2$ . For  $\alpha \in [h_0^{-1}, 1/2]$  it is easy to see that the right-hand side of (14) is  $\ll h_0 \alpha^{-1}$ . Summing up, we have

$$\frac{d}{d\alpha} \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \ll \min(\alpha h_0^3, \alpha^{-1} h_0). \quad (15)$$

A straightforward computation reveals that

$$\frac{d}{d\alpha} \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \right)^2 = 2 \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \frac{d}{d\alpha} \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \ll \min(\alpha h_0^4, \alpha^{-2} h_0),$$

by (15) and (13). Furthermore

$$\begin{aligned} \frac{d}{d\alpha} \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \cos(\pi(h_0 + 1)\alpha) \right) &= \frac{d}{d\alpha} \left( \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \right) \cos(\pi(h_0 + 1)\alpha) \\ &\quad - \pi(h_0 + 1) \frac{\sin(\pi h_0 \alpha)}{\sin(\pi \alpha)} \sin(\pi(h_0 + 1)\alpha), \end{aligned}$$

and a similar computation yields

$$B(\alpha, h) \ll \min(\alpha h_0^3, \alpha^{-1} h_0),$$

which is of lower order of magnitude. Hence the second part of Lemma 2 is proved.  $\square$

We also remark that estimates similar to the ones in Lemma 2 hold for  $U(\alpha, h)$  too; since they immediately follow from the definition we omit their proofs.

Let now

$$\widehat{f}(t) = \int_{-\infty}^{+\infty} f(x) e(-tx) dx$$

be the Fourier transform of  $f(x)$ . We need the following auxiliary result which is based on Gallagher's lemma.

**Lemma 3.** *Let  $0 < h \leq X$ ,*

$$R(\alpha) = S(\alpha) - T(\alpha) \quad \text{and} \quad \widetilde{R}(\alpha) = \widetilde{S}(\alpha) - \widetilde{T}(\alpha). \quad (16)$$

*Then*

$$\int_{-1/2}^{1/2} |R(\alpha)|^2 K(\alpha, h) d\alpha = \int_{-\infty}^{+\infty} |\widetilde{R}(\alpha)|^2 U(\alpha, h) d\alpha = J(X, h) + \mathcal{O}(E(X, h)),$$

*where  $E(X, h)$  is defined in (7). Moreover we have,*

$$\int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2 K(\alpha, h) d\alpha = \int_{-\infty}^{+\infty} |\widetilde{R}(\alpha)|^2 U(\alpha, h) d\alpha = \widetilde{J}(X, h) + \mathcal{O}(\widetilde{E}(X, h)),$$

where, for every fixed  $\varepsilon > 0$ , we define

$$\tilde{E}(X, h) = \begin{cases} (h+1)^3(\log X)^2 & (\text{uncond.}) \text{ uniformly for } 0 < h \leq X^\varepsilon \\ h^3 & (\text{uncond.}) \text{ uniformly for } X^\varepsilon < h \leq X \\ (h+1)^2(\log X)^4 & (\text{under RH}) \text{ uniformly for } 0 < h \leq X. \end{cases} \quad (17)$$

**Proof.** The first part is Lemma 1 of [6], so we skip the proof. For the second part, we start remarking that Lemma 1.9 of Montgomery [8] gives

$$\int_{-\infty}^{+\infty} |\tilde{R}(\alpha)|^2 U(\alpha, h) d\alpha = \int_{-\infty}^{+\infty} \left| \sum_{\substack{|n-x| < h/2 \\ n \geq 1}} (\Lambda(n) - 1) e^{-n/X} \right|^2 dx. \quad (18)$$

By periodicity we have

$$\int_{-\infty}^{+\infty} |\tilde{R}(\alpha)|^2 U(\alpha, h) d\alpha = \int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 \left( \sum_{n=-\infty}^{+\infty} U(n + \alpha, h) \right) d\alpha.$$

Since  $\hat{U}(\alpha, h) = \max(h - |\alpha|; 0)$ , by Poisson's summation formula and (10) we get

$$\sum_{n=-\infty}^{+\infty} U(n + \alpha, h) = \sum_{n=-\infty}^{+\infty} \hat{U}(\alpha, h) e(n\alpha) = K(\alpha, h),$$

and hence, using (18), we obtain

$$\begin{aligned} \int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) d\alpha &= \int_{-\infty}^{+\infty} \left| \sum_{\substack{x < n \leq x+h \\ n \geq 1}} (\Lambda(n) - 1) e^{-n/X} \right|^2 dx \\ &= \int_0^{+\infty} \left| \sum_{x < n \leq x+h} (\Lambda(n) - 1) e^{-n/X} \right|^2 dx + \mathcal{O}((h+1)^2(\log(h+1))^4), \end{aligned} \quad (19)$$

where in the last estimate we assumed RH and we used

$$\psi(y) = y + \mathcal{O}(y^{1/2}(\log y)^2) \quad (20)$$

on a interval of length  $\leq h$ . Noting that

$$\sum_{x < n \leq x+h} (\Lambda(n) - 1) e^{-n/X} = e^{-x/X} (\psi(x+h) - \psi(x) - h) \left( 1 + \mathcal{O}\left(\frac{h+1}{X}\right) \right)$$

and recalling that  $h \leq X$ , from (19) we have

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) d\alpha = \tilde{J}(X, h) \left( 1 + \mathcal{O}\left(\frac{h+1}{X}\right) \right) + \mathcal{O}((h+1)^2(\log X)^4).$$

To estimate the last error term we connect  $\tilde{J}(X, h)$  to  $J(X, h)$ . A partial integration immediately gives

$$\tilde{J}(X, h) = \frac{2}{X} \int_0^\infty J(t, h) e^{-2t/X} dt. \quad (21)$$

To estimate the right-hand side of (21), we split the range of integration into  $[0, h] \cup [h, +\infty)$ . A direct computation using (20) shows that

$$\int_0^h J(t, h) e^{-2t/X} dt \ll h(\log h)^4 \int_0^h t e^{-2t/X} dt \ll h^3(\log h)^4.$$

Still assuming RH, the Selberg [10] estimate gives, for  $1 \leq h \leq t$ , that

$$J(t, h) \ll ht(\log t)^2 \quad (22)$$

and so we get

$$\int_h^{+\infty} J(t, h) e^{-2t/X} dt \ll h \int_h^{+\infty} t(\log t)^2 e^{-2t/X} dt \ll hX^2(\log X)^2.$$

Summing up, under RH we have

$$\tilde{J}(X, h) \ll (h+1)X(\log X)^4$$

we can finally write

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) d\alpha = \tilde{J}(X, h) + \mathcal{O}((h+1)^2(\log X)^4).$$

The unconditional cases follow by replacing (20) with the Brun-Titchmarsh inequality and (22) with the Lemma in [5].  $\square$

In the next sections we will also need the following remark. Let  $\xi > 0$  and  $\delta\xi = 1/2$ . In this case  $U(\alpha, \delta) \gg \delta^2$  for  $|\alpha| \leq \xi$ ; hence by the first equation in Lemma 3 we obtain

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 d\alpha \ll \xi^2 \left( J\left(X, \frac{1}{2\xi}\right) + E\left(X, \frac{1}{2\xi}\right) \right). \quad (23)$$

By (22) and (7), under RH we immediately obtain, for every  $1/(2X) \leq \xi \leq 1/2$ , that

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 d\alpha \ll X\xi(\log X)^4. \quad (24)$$

### 3. PROOF OF THEOREM 1

We use Lemma 3 in the form

$$J(X, h) = \int_{-1/2}^{1/2} |R(\alpha)|^2 K(\alpha, h) d\alpha + \mathcal{O}(E(X, h)), \quad (25)$$

where  $R(\alpha)$  is defined in (16). Observe that both  $|R(\alpha)|^2$  and  $K(\alpha, h)$  are even functions of  $\alpha$ , and hence we may restrict our attention to  $\alpha \in [0, 1/2]$ . Writing

$$f(X, \alpha) = X \log(X\alpha) + \left(\frac{c}{2} + 1\right)X = X \log \frac{X}{h} + X \log(h\alpha) + \left(\frac{c}{2} + 1\right)X, \quad (26)$$

we can approximate  $|R(\alpha)|^2$  as  $|R(\alpha)|^2 = f(X, \alpha) + (|R(\alpha)|^2 - f(X, \alpha))$ . Using Lemma 1 and (26), we obtain

$$\int_0^{1/2} f(X, \alpha) K(\alpha, h) d\alpha = \frac{h}{2} X \log \frac{X}{h} + c' \frac{h}{2} X + \mathcal{O}(X), \quad (27)$$

where  $c'$  is defined in (5).

Let now  $U_1 < 1/h < U_2 \leq 1$  be two parameters to be chosen later. Hence by Lemma 2 and (24) we immediately obtain

$$\begin{aligned} \int_0^{U_1} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha &\ll h^2 \int_0^{U_1} |R(\alpha)|^2 d\alpha + h^2 \int_0^{U_1} f(X, \alpha) d\alpha \\ &\ll h^2 U_1 X (\log X)^4. \end{aligned} \quad (28)$$



Again by Lemma 2 and (24), by partial integration we have

$$\begin{aligned} \int_{U_2}^{1/2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha &\ll \int_{U_2}^{1/2} \frac{|R(\alpha)|^2}{\alpha^2} d\alpha + \int_{U_2}^{1/2} \frac{f(X, \alpha)}{\alpha^2} d\alpha \\ &\ll \frac{X(\log X)^4}{U_2}. \end{aligned} \quad (29)$$

From (28)-(29) it is clear that the optimal choice is  $h^2 U_1 = 1/U_2$ . We now evaluate

$$\int_{U_1}^{U_2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha.$$

A direct computation and the hypothesis show that

$$\int_0^\xi (|R(\alpha)|^2 - f(X, \alpha)) d\alpha \ll \frac{(X\xi)^{1-a}}{(\log X\xi)^b},$$

and hence, by partial integration and Lemma 2, we obtain

$$\begin{aligned} \int_{U_1}^{U_2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha &\ll h^2 \frac{(XU_1)^{1-a}}{(\log X)^b} + \frac{X^{1-a} U_2^{-1-a}}{(\log X)^b} \\ &\quad + \frac{hX^{1-a}}{(\log X)^b} \int_{U_1}^{U_2} \xi^{2-a} \min(h^3, \xi^{-3}) d\xi. \end{aligned}$$

Using the constraints  $h^2 U_1 = 1/U_2$  and  $U_1 < 1/h$ , the right-hand side is

$$\ll \frac{h^{1+a} X^{1-a}}{(\log X)^b} + \frac{hX^{1-a}}{(\log X)^b} \int_{1/h}^{U_2} \xi^{-1-a} d\xi \ll \frac{h^{1+a} X^{1-a}}{(\log X)^b} + R_{a,b}(X, h, U_2),$$

where

$$R_{a,b}(X, h, U_2) = \begin{cases} hX \log(hU_2) (\log X)^{-b} & \text{if } a = 0 \\ h^{1+a} X^{1-a} (\log X)^{-b} & \text{if } a > 0. \end{cases}$$

Combining such results we get

$$\int_{U_1}^{U_2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha \ll R_{a,b}(X, h, U_2). \quad (30)$$

Hence, by (28)-(30) and  $h^2 U_1 = 1/U_2$  we get

$$\int_0^{1/2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) d\alpha \ll \frac{X(\log X)^4}{U_2} + R_{a,b}(X, h, U_2). \quad (31)$$

Choosing

$$U_2 = \frac{X^a (\log X)^{b+4}}{h^{1+a}} \quad \text{and} \quad U_1 = \frac{h^{a-1}}{X^a (\log X)^{b+4}},$$

by (27) and (31) we finally get

$$\int_0^{1/2} |R(\alpha)|^2 K(\alpha, h) d\alpha = \frac{h}{2} X \log \frac{X}{h} + c' \frac{h}{2} X + \mathcal{O}(X + R_{a,b}(X, h))$$

where  $c'$  and  $R_{a,b}(X, h)$  are defined in (5) and (6). Theorem 1 follows from (25).

#### 4. PROOF OF THEOREM 2

We adapt the proof of Lemma 5 of [7] (which is an explicit form of Lemma 4 of [2]). We recall that  $0 < \eta < 1/4$  is a parameter to be chosen later and

$$K_\eta(x) = \frac{\sin(2\pi x) + \sin(2\pi(1+\eta)x)}{2\pi x(1-4\eta^2 x^2)},$$

so that

$$\widehat{K}_\eta(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ \cos^2\left(\frac{\pi(|t|-1)}{2\eta}\right) & \text{if } 1 \leq |t| \leq 1+\eta \\ 0 & \text{if } |t| \geq 1+\eta \end{cases}$$

and

$$K_\eta''(x) \ll \min(1; (\eta x)^{-3}), \quad (32)$$

see Eqs. (3.14)-(3.15) and Lemma 4 of [7]. Moreover, by Lemma 3 of [7], we also have

$$\widehat{K}_\eta(t) = \int_0^\infty K_\eta''(x)U(t, x) dx. \quad (33)$$

Hence, again considering only positive values of  $\alpha$ , we have

$$\int_0^\infty |R(\alpha)|^2 \widehat{K}_\eta\left(\frac{\alpha}{\xi}(1+\eta)\right) d\alpha \leq \frac{R(X, \xi)}{2} \leq \int_0^\infty |R(\alpha)|^2 \widehat{K}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha \quad (34)$$

where  $R(\alpha)$  is defined in (16). Writing  $f(X, \alpha)$  as in (26), we approximate  $|R(\alpha)|^2$  as  $|R(\alpha)|^2 = f(X, \alpha) + (|R(\alpha)|^2 - f(X, \alpha))$ . Observing that  $U(\alpha/\xi, x) = \xi^2 U(\alpha, x/\xi)$ , letting

$$g(x, \xi) = \xi^2 \int_0^\infty (|R(\alpha)|^2 - f(X, \alpha))U\left(\alpha, \frac{x}{\xi}\right) d\alpha$$

and using (33), we get

$$\int_0^\infty |R(\alpha)|^2 \widehat{K}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha = \int_0^\infty f(X, \alpha) \widehat{K}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha + \int_0^\infty K_\eta''(x)g(x, \xi) dx = J_1 + J_2, \quad (35)$$

say. A direct computation shows that

$$J_1 = X\xi \log X\xi + \frac{c}{2}X\xi + \mathcal{O}(\eta X\xi \log X\xi). \quad (36)$$

In order to estimate  $J_2$  we first remark that by Lemma 1, (26) and (5), we have

$$\xi^2 \int_0^\infty f(X, \alpha)U\left(\alpha, \frac{x}{\xi}\right) d\alpha = \frac{xX\xi}{2} \log \frac{X\xi}{x} + \frac{c'}{2}xX\xi. \quad (37)$$

Now we need the following

**Lemma 4.** *Assume RH and let  $\varepsilon > 0$ . We have*

$$g(x, \xi) \ll \begin{cases} X\xi^2 \log X & \text{if } 0 < x \leq \xi \\ xX\xi(\log X)^2 & \text{if } \xi \leq x \leq \xi X^{1/2-\varepsilon} \\ xX\xi(\log X)^4 & \text{if } x \geq \xi X^{1/2-\varepsilon}. \end{cases} \quad (38)$$

Assume further the hypothesis of Theorem 2. We have

$$g(x, \xi) \ll x^{1+a} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi} (\log X)^2 \quad \text{if } \xi \leq x \leq \xi X^{1/2-\varepsilon}. \quad (39)$$

**Proof.** Let  $h = x/\xi$ . We first prove (38). For  $0 < h \leq 1$  we have  $U(\alpha, h) \ll \min(1; \alpha^{-2})$  and hence by periodicity

$$\int_0^\infty |R(\alpha)|^2 U(\alpha, h) d\alpha \ll \sum_{n=1}^\infty \frac{1}{n^2} \int_{n-1}^n |R(\alpha)|^2 d\alpha \ll T\left(X, \frac{1}{2}\right) + S\left(X, \frac{1}{2}\right) \ll X \log X,$$

by the Prime Number Theorem (this case is, in fact, unconditional). For  $1 \leq h \leq X^{1/2-\varepsilon}$  the assertion follows immediately from (23), (22), (7) and (37). Finally, for  $h \geq X^{1/2-\varepsilon}$  we use (20) after having applied Gallagher's lemma, see Lemma 1.9 of Montgomery [8], which gives

$$\int_{-\infty}^{+\infty} |R(\alpha)|^2 U(\alpha, h) d\alpha = \int_{-\infty}^{+\infty} \left| \sum_{\substack{|n-x| < h/2 \\ 1 \leq n \leq X}} (\Lambda(n) - 1) \right|^2 dx.$$

Using (37), this case holds true. We now prove (39). For  $1 \leq h \leq X^{1/2-\varepsilon}$  the assertion follows immediately from (37), Lemma 3 and the hypothesis of Theorem 2.  $\square$

Choosing now  $V_1, V_2$  such that  $\xi < V_1 < 1/\eta < V_2 < \xi X^{1/2-\varepsilon}$ , we split  $J_2$ 's integration range into six subintervals. We obtain

$$\begin{aligned} J_2 &= \left( \int_0^\xi + \int_\xi^{V_1} + \int_{V_1}^{1/\eta} + \int_{1/\eta}^{V_2} + \int_{V_2}^{\xi X^{1/2-\varepsilon}} + \int_{\xi X^{1/2-\varepsilon}}^{+\infty} \right) K''_\eta(x) g(x, \xi) dx \\ &= M_1 + M_2 + M_3 + M_4 + M_5 + M_6, \end{aligned} \quad (40)$$

say. By Lemma 4 and (32), we obtain

$$M_1 \ll X \xi^2 \log X \int_0^\xi dx \ll X \xi^3 \log X,$$

$$M_2 \ll X \xi (\log X)^2 \int_\xi^{V_1} x dx \ll X \xi V_1^2 (\log X)^2,$$

$$M_3 \ll \int_{V_1}^{1/\eta} \left( x^{1+a} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi} (\log X)^2 \right) dx \ll \frac{(X\xi)^{1-a}}{\eta^{2+a} (\log X)^b} + \frac{(\log X)^2}{\xi \eta^4},$$

$$M_4 \ll \frac{1}{\eta^3} \int_{1/\eta}^{V_2} \left( x^{a-2} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{(\log X)^2}{\xi} \right) dx \ll \frac{(X\xi)^{1-a}}{\eta^{2+a} (\log X)^b} + \frac{V_2 (\log X)^2}{\xi \eta^3},$$

$$M_5 \ll \frac{X \xi (\log X)^2}{\eta^3} \int_{V_2}^{\xi X^{1/2-\varepsilon}} \frac{dx}{x^2} \ll \frac{X \xi (\log X)^2}{V_2 \eta^3},$$

and

$$M_6 \ll \frac{X \xi (\log X)^4}{\eta^3} \int_{\xi X^{1/2-\varepsilon}}^{+\infty} \frac{dx}{x^2} \ll \frac{X^{1/2+\varepsilon} (\log X)^4}{\eta^3}.$$

Hence, recalling  $\xi > X^{-1/2+\varepsilon}$ , by (40) and the definitions of  $V_1$  and  $V_2$  we get

$$J_2 \ll X \xi (\log X)^2 \left( V_1^2 + \frac{(\log X)^2}{V_2 \eta^3} \right) + \frac{(X\xi)^{1-a}}{\eta^{2+a} (\log X)^b}. \quad (41)$$

Choosing  $V_1 = \eta^{1/2}/\log X$  and  $V_2 = \log^3 X/\eta^4$ , by (35)-(36) and (41), we obtain

$$\int_0^\infty |R(\alpha)|^2 \widehat{K}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha = X \xi \log X \xi + \frac{c}{2} X \xi + \mathcal{O}\left(\eta X \xi \log X + \frac{(X\xi)^{1-a}}{\eta^{2+a} (\log X)^b}\right). \quad (42)$$

To optimize the error term we choose  $\eta^{3+a} = (X\xi)^{-a}(\log X)^{-b-1}$ , so that (42) becomes

$$\int_0^\infty |R(\alpha)|^2 \widehat{K}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha = X\xi \log X\xi + \frac{c}{2}X\xi + \mathcal{O}\left(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\right). \quad (43)$$

Finally, by (34) and (43), we obtain

$$R(X, \xi) \leq 2X\xi \log X\xi + cX\xi + \mathcal{O}\left(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\right).$$

In a similar way we also get that

$$R(X, \xi) \geq 2X\xi \log X\xi + cX\xi + \mathcal{O}\left(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\right),$$

and Theorem 2 follows.

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